MIDTERM

M559 – LINEAR ALGEBRA

1. For an $n \times n$ matrix $X = [x_{i,j}]$, we define the *trace* of X, denoted by $\mathrm{tr}(X)$, to be the sum of the elements in the main diagonal, i.e.,

$$\mathrm{tr}(X) \stackrel{\text{def}}{=} \sum_{i=1}^{n} x_{i,i}.$$

Let $A = [a_{i,j}]$ and $B = [b_{i,j}]$ be $n \times n$ matrices.

(a) Prove that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

(b) Prove that if P is an invertible matrix, then $\mathrm{tr}(A) = \mathrm{tr}(P^{-1}AP)$.
[You may use part (a) here even if you could not do it.]

Proof. (a) We have:

$$\begin{aligned}
\mathrm{tr}(AB) &= \sum_{i=1}^{n} \sum_{k=1}^{n} a_{i,k} b_{k,i} \\
&= \sum_{k=1}^{n} \sum_{i=1}^{n} a_{i,k} b_{k,i} && \text{[switch the sum]} \\
&= \sum_{k=1}^{n} \sum_{i=1}^{n} b_{k,i} a_{i,k} && \text{[switch factors]} \\
&= \sum_{i=1}^{n} \sum_{k=1}^{n} b_{i,k} a_{k,i} && \text{[reindex: } (k,i) \mapsto (i,k)\text{]} \\
&= \mathrm{tr}(BA).
\end{aligned}$$

(b) We have, using (a), that

$$\mathrm{tr}(A) = \mathrm{tr}(P(P^{-1}A)) = \mathrm{tr}((P^{-1}A)P) = \mathrm{tr}(P^{-1}AP).$$

□

2. Let V be an n-dimensional $\mathbb{C}$-vector space [n finite] and W be a subspace of V, and let

$$W^0 = \{T \in \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}) \,:\, T(w) = 0 \text{ for all } w \in W\} \leq \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}).$$

[Note that W^0 is a subspace of $\mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}) = L(V, \mathbb{C})$, and not V or W.] Prove that

$$\dim W^0 = n - \dim W.$$

Hints:

(1) Start with a basis $\{v_1, \ldots, v_m\}$ of W and complete to a basis $\{v_1, \ldots, v_{m+1}, \ldots, v_n\}$ of V.

(2) Find basis elements $T_{m+1}, T_{m+2}, \ldots, T_n$ for W^0 using the basis of V above.

Proof. Let $\{v_1, \ldots, v_m\}$ be a basis of W and complete it to a basis of V, say,

$$\mathcal{B} = \{v_1, \ldots, v_m, v_{m+1}, \ldots, v_n\}.$$

For $i = m+1, m+2, \ldots, n$ and $j = 1, 2, \ldots, n$ define

$$T_i(v_j) \overset{\text{def}}{=} \delta_{i,j} = \begin{cases} 1, & \text{if } j = i; \\ 0, & \text{if } j \neq i. \end{cases}$$

So, for any $i \in \{m+1, \ldots, n\}$ and $j \in \{1, 2, \ldots, m\}$, we have $T_i(v_j) = 0$, and hence $T_i \in W^0$.

If

$$a_{m+1}T_{m+1} + \cdots + a_nT_n = 0,$$

then, for all $i \in \{m+1, \ldots, n\}$, evaluating at v_i we get $a_i = 0$, so $\mathcal{C} \overset{\text{def}}{=} \{T_{m+1}, \ldots, T_n\}$ is linearly independent.

Now, given $T \in W^0$ and for $i \in \{m+1, \ldots, n\}$, let $a_i \overset{\text{def}}{=} T(v_i)$, and define $S \overset{\text{def}}{=} a_{m+1}T_{m+1} + \cdots + a_nT_n$. Then, we have that, for $i \leq m$:

$$T(v_i) = 0 = \sum_{j=m+1}^{n} a_j \cdot 0 = \sum_{j=m+1}^{n} a_jT_j(v_i) = S(v_i).$$

For $m+1 \leq i \leq n$, we have

$$T(v_i) = a_i = a_iT_i(v_i) = \sum_{j=m+1}^{n} a_jT_j(v_i) = S(v_i).$$

Hence, $T(v_i) = S(v_i)$ for $i = 1, \ldots, n$, so $T = S \in \mathrm{span}(\mathcal{C})$.

Therefore, $\mathcal{C}$ is a basis of W^0 and $\dim W^0 = n - m$. □

3. Let $B \in M_n(\mathbb{C})$ [some *fixed* matrix] and define $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ by $T(A) = BA$. [You may assume without proof that T is linear, as it is pretty clear.] Prove that $\mu_T = \mu_B$, i.e., that the minimal polynomials of the matrix B and of the linear operator T are the same.

Hints:

(1) What is $T^k(A)$?

(2) If $X \in M_n(\mathbb{C})$, then $X = 0$ if and only if $XA = 0$ for all $A \in M_n(\mathbb{C})$. [You can use this without proving.]

(3) Prove that if $f \in \mathbb{C}[x]$, then $f(T) = 0$ if and only if $f(B) = 0$.

Proof. First, note that

$$T^k(A) = T^{k-1}(BA) = T^{k-2}(B^2A) = \cdots = B^kA.$$

Now, let $f = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \in \mathbb{C}[x]$. Then,

$$\begin{aligned} f(T)(A) &= (a_nT^n + a_{n-1}T^{n-1} + \cdots + a_1T + a_0I)(A) \\ &= a_nT^n(A) + a_{n-1}T^{n-1}(A) + \cdots + a_1T(A) + a_0A \\ &= a_nB^nA + a_{n-1}B^{n-1}A + \cdots + a_1BA + a_0A \\ &= (a_nB^n + a_{n-1}B^{n-1} + \cdots + a_1B + a_0) \cdot A \\ &= f(B) \cdot A. \end{aligned}$$

Hence, $f(T) = 0$ if and only if $f(T)(A) = 0$ for all $A \in M_n(F)$ if and only if $f(B)A = 0$ for all $A \in M_n(F)$ if and only if $f(B) = 0$.

So, the annihilators of T and B are the same, and hence their minimal polynomial [the monic polynomial of smallest degree in the annihilator] are the same. □