MIDTERM

M559 – LINEAR ALGEBRA

1. For an $n \times n$ matrix $X = [x_{i,j}]$, we define the *trace* of X, denoted by $\mathrm{tr}(X)$, to be the sum of the elements in the main diagonal, i.e.,

$$\mathrm{tr}(X) \stackrel{\text{def}}{=} \sum_{i=1}^{n} x_{i,i}.$$

Let $A = [a_{i,j}]$ and $B = [b_{i,j}]$ be $n \times n$ matrices.

(a) Prove that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

(b) Prove that if P is an invertible matrix, then $\mathrm{tr}(A) = \mathrm{tr}(P^{-1}AP)$.

[You may use part (a) to prove part (b) even if you could not do it.]

2. Let V be an n-dimensional $\mathbb{C}$-vector space [n finite] and W be a subspace of V, and let

$$W^0 = \{T \in \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}) \, : \, T(w) = 0 \text{ for all } w \in W\} \leq \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}).$$

[Note that W^0 is a subspace of $\mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}) = L(V, \mathbb{C})$, and not V or W.] Prove that

$$\dim W^0 = n - \dim W.$$

3. Let $B \in M_n(\mathbb{C})$ [some *fixed* matrix] and define $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ by $T(A) = BA$. [You may assume without proof that T is linear, as it is pretty clear.] Prove that $\mu_T = \mu_B$, i.e., that the minimal polynomials of the matrix B and of the linear operator T are the same.

Hint: Prove that if $f \in \mathbb{C}[x]$, then $f(T) = 0$ if and only if $f(B) = 0$.