FINAL

M559 – LINEAR ALGEBRA – MAY 8TH, 2026

All vector spaces are assumed to be finite-dimensional, over the complex numbers, and all matrices are assumed to be complex.

1. Let S and T be linear transformations from $\mathbb{C}^6$ to $\mathbb{C}^2$. Prove that there is $v \in \mathbb{C}^6 \setminus \{\vec{0}\}$ such that $S(v) = T(v) = \vec{0}$.

Proof. We have that $\operatorname{rank}(S), \operatorname{rank}(T) \leq 2$, so $\operatorname{null}(S), \operatorname{null}(T) \geq 4$. Hence $\dim(\ker(S) \cap \ker(T)) > 0$, as otherwise

$$\begin{aligned} 6 &= \dim \mathbb{C}^6 \\ &\geq \dim \ker(S) + \ker(T) \\ &= \dim \ker(S) + \dim \ker(T) - \dim(\ker(S) \cap \ker(T)) \\ &\geq 4 + 4 = 8, \end{aligned}$$

a contradiction. □

2. Given examples of two real 4×4 *nilpotent* matrices that have the same minimal and characteristic polynomials, but are not similar. **Justify!**

Solution. We have that

$$A \overset{\text{def}}{=} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad B \overset{\text{def}}{=} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

are both in Jordan form, from which we get that $\chi_A = \chi_B = x^4$, so both are nilpotent, and from the first 2×2 elementary Jordan block, we get that $\mu_A = \mu_B = x^2$, but since they have different Jordan forms they are not similar [or note that the first has rank 2 and the second has rank 1]. ☐

3. Let A be a 3×3 matrix over $\mathbb{R}$ with eigenvalues -1, 1, and 2, and $B \overset{\text{def}}{=} A^3 - 3A + I$. Compute $\det(B)$.

Proof. Since the matrix is 3×3 and we have three distinct eigenvalues we have that there $P \in \mathrm{GL}_3(\mathbb{R})$ such that such that

$$PAP^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Hence,

$$\begin{aligned} PBP^{-1} &= P(A^3 - 3A + I)P^{-1} \\ &= PA^3P^{-1} - 3PAP^{-1} + I \\ &= (PAP^{-1})^3 - 3PAP^{-1} + I \\ &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 8 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -6 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}. \end{aligned}$$

So,

$$\det(B) = \det(PBP^{-1}) = 3 \cdot (-1) \cdot 3 = -9.$$

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4. Let U be an $n \times n$ unitary matrix. Show that $|\det U| = 1$.

Proof. We have that

$$\det(U^*) = \det\left(\overline{U^t}\right) = \overline{\det\left(U^t\right)} = \overline{\det(U)}.$$

On the other hand

$$1 = \det(I) = \det(UU^*) = \det(U) \cdot \det(U^*) = \det(U)\overline{\det(U)} = |\det(U)|^2.$$

Hence, $|\det(U)| = 1$. ☐

5. Let V be a finite dimensional vector space over $\mathbb{C}$ and T be a Hermitian operator on V. Prove that if there is $v \in V$ with $\|v\| = 1$ and $c \in \mathbb{C}$ such that

$$\|T(v) - cv\| < \epsilon$$

for some $\epsilon > 0$, then there is $c' \in \operatorname{spec}(T)$ such that $|c - c'| < \epsilon$.

Proof. Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be a basis of eigenvectors, with $T(v_i) = c_i v_i$. Suppose that for all i we have $|c - c_i| \geq \epsilon$. Then, if $v = a_1 v_1 + \cdots + a_n v_n$, we have

$$\begin{aligned}
\|T(v) - cv\|^2 &= \left\| \sum_{i=1}^{n} a_i (c_i - c) v_i \right\|^2 \\
&= \sum_{i=1}^{n} |a_i|^2 |c_i - c|^2 \\
&\geq \sum_{i=1}^{n} |a_i|^2 \cdot \epsilon^2 \\
&= \epsilon^2 \cdot \sum_{i=1}^{n} |a_i|^2 \\
&= \epsilon^2,
\end{aligned}$$

a contradiction. ☐

6. Let T be a normal operator on a finite dimensional vector space V with $\operatorname{spec}(T) \subseteq \mathbb{R}$. Prove that T is Hermitian.

Proof. Since T is normal, we have an orthonormal basis $\mathcal{B}$ of eigenvectors, so $[T]_{\mathcal{B}}$ is diagonal with entries in $\mathbb{R}$, and hence $[T]_{\mathcal{B}}^* = [T]_{\mathcal{B}}$. But then, since $\mathcal{B}$ is orthonormal, we have $[T^*]_{\mathcal{B}} = [T]_{\mathcal{B}}^* = [T]_{\mathcal{B}}$, and hence $T = T^*$. ☐