FINAL

M559 – LINEAR ALGEBRA – MAY 8TH, 2026

All vector spaces are assumed to be finite-dimensional, over the complex numbers, and all matrices are assumed to be complex.

1. Let S and T be linear transformations from $\mathbb{C}^6$ to $\mathbb{C}^2$. Prove that there is $v \in \mathbb{C}^6 \setminus \{\vec{0}\}$ such that $S(v) = T(v) = \vec{0}$.

2. Given examples of two real 4×4 *nilpotent* matrices that have the same minimal and characteristic polynomials, but are not similar. **Justify!**

3. Let A be a 3×3 matrix over $\mathbb{R}$ with eigenvalues -1, 1, and 2, and $B \overset{\text{def}}{=} A^3 - 3A + I$. Compute $\det(B)$.

4. Let U be an $n \times n$ unitary matrix. Show that $|\det U| = 1$.

5. Let V be a finite dimensional vector space over $\mathbb{C}$ and T be a Hermitian operator on V. Prove that if there is $v \in V$ with $\|v\| = 1$ and $c \in \mathbb{C}$ such that

$$\|T(v) - cv\| < \epsilon$$

for some $\epsilon > 0$, then there is $c' \in \text{spec}(T)$ such that $|c - c'| < \epsilon$.

6. Let T be a normal operator on a finite dimensional vector space V with $\text{spec}(T) \subseteq \mathbb{R}$. Prove that T is Hermitian.